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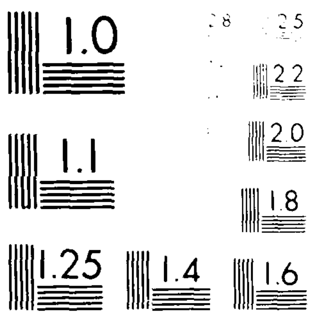
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Report 2336

NEW RADIUS OF CURVATURE FORMULAE AND THEIR APPLICATIONS

by
Melvin H. Friedman, PhD

September 1981

U.S. ARMY MOBILITY EQUIPMENT RESEARCH AND DEVELOPMENT COMMAND
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The shape of a surface influences reflectance properties; therefore, shape is one factor among many which determines how difficult it is to "see" a surface-laid mine from the air. The shape of the surface can be locally characterized using the radius of curvature concept. A survey of the literature showed no systematic radius of curvature treatment which displayed explicit formulae for calculating radius of curvature in three dimensions and which emphasized applications of this concept. In this report a new and physically motivated definition for radius of curvature is given and from this definition the well-known and

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some hitherto unknown radius of curvature formulae are derived. The problem of computing radius of curvature in three dimensions for curves known at discrete points is solved and applied to the problem of measuring one parameter characterizing enemy aircraft maneuverability from radar data. Other applications include: the design of a device for measuring road or railroad track radius of curvature, the design of a device for checking highspeed roads for proper banking and a technique for the use of radius of curvature concepts in the design of a land navigational system.

SUMMARY

A definition for radius of curvature motivated by well-known geometrical properties of a circle and different from the usual definition for radius of curvature is given. The new definition is used to derive the usual two dimensional formulae for radius of curvature in cartesian and polar coordinates. The usual parametric formula for the radius of curvature in a two dimensional cartesian coordinate system is also derived, cast into vector form, and then physical arguments are used to show that the vector formula for radius of curvature is valid for space curves. By use of this result and well-known kinematic expressions for the acceleration and velocity in three dimensional coordinate systems, parametric formulae for the radius of curvature in rectangular, cylindrical, and spherical coordinates are derived. These formulae are then used to derive equations for the radius of curvature in these coordinate systems when two of the variables are known as functions of the third variable and also for the case where the variables are implicitly related to each other. The question of computing the radius of curvature for space curves defined only at discrete points is considered, and an explicit formula suitable for numerical calculation under these conditions is derived. The results of this work are then applied to four problems:

a. The measurement of one parameter characterizing enemy aircraft maneuverability from radar data.

b. The design of a device for measurement of radius of curvature for roads or railroad track.

c. The design of a device for checking highspeed roads for proper banking.

d. A technique for the possible use of radius of curvature concepts in the design of a land navigational system.

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ILLUSTRATIONS

NEW RADIUS OF CURVATURE FORMULAE AND THEIR APPLICATIONS

I. INTRODUCTION

1. Introduction. The concept of radius of curvature is useful in many applications. Because the focal length of a lens or mirror depends on the radius of curvature associated with the defining surface,[1] researchers are interested in measuring and controlling the radius of curvature in biological[2-4] and non-biological[5-7] optical systems. Bent crystals are used in electron probe microanalysis,[8] and for this reason and others, there is an interest in measuring[9-11] the curvature of crystals. The concept has found application in the theory of linkages,[12-13] in gear design,[14] in rotor vibration,[15] in flux containment,[16] and in bone prostesis.[17] To date, the radius of curvature concept does not appear to have been used in mine detection, but this may change.

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[1] E. Hecht and A. Zajac, *Optics*, Addison-Wesley Publishing Co. (1976).

[2] W. Steel and D. Noack, *Measuring Radius of Curvature of Soft Corneal Lenses*, Applied Optics, *16*, N4, 778 (1977).

[3] R. D. Freeman, *Corneal Radius of Curvature of the Kitten and Cat*, Investigative Ophthalmology and Visual Science, *19*, N3, 306 (1980).

[4] C. S. Maclatchy, *Radius of Curvature of the Cornea — An Experiment for the Life Science Physics Lab*, Amer. Jour. of Physics, *46*, N6, 615 (1978).

[5] M. C. Gerchman and G. C. Hunter, *Differential technique for Accurately Measuring the Radius of Curvature of Long Radius Concave Optical Surfaces*, Opt. Eng. *19*, N6, 843 (1980).

[6] P. D. McGrath, *Volume Method for Finding Radius of Curvature*, Sky and Telescope, *59*, 71 (1980).

[7] V. I. Sherstobitov, *Cylindrical Mirror With a Controlled Radius of Curvature*, Sov. J. Quantum Electron., *4*, (12)m K471 (1975).

[8] L. S. Birks, *Electron Probe Microanalysis*, John Wiley and Sons, Inc. (1971).

[9] E. Zschech, G. Merz, W. Blau and K. Kleinstock, *A Simple Method for Determining the Radius of Curvature of Bent Spectrometer Crystals*, Kristall und Technik- Crystal Research and Technology, *15*, N3, 25 (1980).

[10] K. Godwod, A. T. Nagy and Z. Rek, *Application of X-ray Triple-Crystal Spectrometer for Measuring Radius of Curvature of Bent Single Crystals*, Physica Status Solidi A-Applied Research *34*, N2, 705 (1976).

[11] V. D. Skupov, G. I. Uspenskaya, *Application of X-ray Method for Precision Measurement of Radius of Curvature in Monocrystalline Plates*, Zavodskaya Laboratoria *41*, N6, 700 (1975).

[12] R. G. Mitchiner and H. H. Mabie, *Synthesis of 4 Bar Linkage Coupler Curves Using Derivatives of Radius of Curvature — Circular Path Procedure*, Mechanism and Machine Theory 12, N2, 147 (1977).

[13] R. G. Mitchiner and H. H. Mabie, *Synthesis of 4 Bar Linkage Coupler Curves Using Derivatives of Radius of Curvature — Straight Path Procedure*, Mechanism and Machine Theory *12*, N2, 133 (1977).

[14] A. P. Bryzbaty, *Tool Curvature Radius and Stress Concentration In Bevel Gear Teeth*, Russian Engineering Journal *54*, N1, 37 (1974).

[15] R. J. Kutcher, *Effect of Radius of Curvature of Ball Bearing Grooves on Rotor Vibration*, Russian Engineering Journal *55*, N1, 13 (1975).

[16] K. Tsuno and K. Nakagawa, *Effect of Radius of Yoke Curvature on Leakage Flux Outside Yoke in Permanent Magnet Assemblies*, Japanese Journal of Applied Physics *15*, N9, 1695 (1976).

[17] F. Johnson, R. G. Burwell, P. H. Dangerfield, R. Mitsud, and H. M. Harrison, *A Computer Technique for Measuring the Radius of Curvature of the Femoral Head*, Journal of Bone and Joint Surgery — British Volume 62, N2, 252 (1980).

The emphasis in mine warfare is shifting away from buried mines that are time consuming to install and remove to surface-laid mines that can be rapidly deployed. Consequently, the emphasis in mine detection is shifting away from closein detection to rapid airborne methods. For the most part, these techniques rely on the reflections of electromagnetic radiation emanating from the surface of the mine. These reflections depend on:

a. The wavelength of the incident and received radiation.

b. The electrical and thermal properties of the mine casing (compared to the surrounding).

c. The shape and finish of the mine surface.

The concept of radius of curvature may prove useful in characterizing mine shapes which determine the reflectance properties and, hence, the detectability of mines.

Direct experimental evidence that the electromagnetic reflected signal from a mine's surface depends on its shape and finish is available to all who have sat in a car in midday waiting for a traffic light to change while observing the reflections from a nearby car. The strongest reflections come from the front or rear windshields. The weakest reflections come from the body of the vehicle, especially, if it is old and the finish was not cared for. These observations are summarized by: the smoothest surfaces give the strongest reflections. One can also observe that the apparent diameter of the sun as seen in different reflecting surfaces changes with the shape of the surface; flat objects give a large apparent diameter while surfaces with sharp curvature give a smaller apparent diameter. When the cars start to move, changing the angles between the observer, the reflector, and the source, one can observe that the conditions which must be satisfied for reflections from a flat object to shine into one's eyes are stringent, while the conditions which must be satisfied from a curved convex object to shine into one's eyes are less stringent. The greater the angle through which the convex surface turns, the less stringent is the condition for the reflection to shine into the eyes of the observer.

The person who would detect mines from the air may like all mines to have convex windshield-like surfaces so that reflections from them would be detected with the ease of reflections from a car during midday in the summer. Of course, the designer of mines does not oblige the person who would detect them. But how close does he come, i.e., how can the influence of shape on the detectability of a surface-laid mine as seen from the air be quantified? To characterize the shape of a mine, something useful for evaluating remote mine field detection methods against different mines, the radius of curvature concept can be used. A survey of the literature[18-32] failed to reveal a systematic radius of curvature treatment useful for calculating this quantity.

One purpose of this report is to give a comprehensive, easy-to-read treatment which emphasizes how to use and how to calculate radius of curvature. Another purpose of this report is to develop the idea of radius of curvature in a more natural and physical way than is usually done.

Given a curve $y = f(x)$, the usual method[18-25] of deriving the formula for the radius of curvature at a point x along the curve starts with the definition of curvature[33] κ

$$\kappa = \left| \frac{d\phi}{ds} \right| \tag{1}$$

where ϕ is the angle between the vector tangent to the curve at the point x and the positive x axis, and s is the curve length measured from an arbitrary point. More advanced treatments[24, 26-28] define curvature by

$$\kappa = \left| \frac{du(s)}{ds} \right| \tag{1}$$

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18 W. Leighton, *Calculus*, Allyn and Bacon, Inc. (1958).

19 T. S. Peterson, *Elements of Calculus*, Harper and Brothers (1950).

20 L. S. Smith, M. Salkover and H. K. Justice, *Calculus*, John Wiley and Sons, Inc. (1958).

21 G. A. Osborne, *Differential and Integral Calculus*, D. C. Heath and Co. (1908).

22 W. A. Granville, P. F. Smith and W. R. Longley, *Elements of the Differential and Integral Calculus*, Ginn and Co. (1934).

23 R. Courant, *Differential and Integral Calculus*, Interscience Publishers, Inc. (1955).

24 G. B. Thomas, *Calculus and Analytic Geometry*, Addison-Wesley Publishing Co. (1968).

25 A. E. Taylor, *Advanced Calculus*, Ginn and Co. (1955).

26 M. M. Lipschutz, *Differential Geometry*, McGraw Hill Book Co. (1969).

27 E. Kreyszig, *Advanced Engineering Mathematics*, John Wiley and Sons, Inc. (1972).

28 W. Kaplan, *Advanced Calculus*, Addison-Wesley (1973).

29 T. M. Apostol, *Mathematical Analysis*, Addison Wesley (1957).

30 T. M. Apostol, *Calculus*, Blaisdell Publishing House (1962).

31 R. C. Buck and E. F. Buck, *Advanced Calculus*, McGraw-Hill, Inc. (1965).

32 K. Rektorys, editor, *Survey of Applicable Mathematics*, The MIT Press (1969).

33 Not all authors take the absolute value.

where $\mathbf{u}$ is a unit vector tangent to the curve and s has the same interpretation as in equation (1) or give[29,30] parametric definitions readily identified as being equivalent to (1). The radius of curvature ρ is then defined by

$$\rho = \frac{1}{\kappa} \ . \tag{2}$$

With these definitions, it is shown in introductory calculus tests that

$$\rho = \frac{(1 + y'^2)^{3/2}}{y''} \ . \tag{3}$$

Typically the formula

$$\rho = \frac{(r^2 + r'^2)^{3/2}}{r^2 + 2r'^2 - rr''} \tag{4}$$

is derived from equation (3) which is applicable to a curve $r = r(\theta)$ given in polar coordinates. Then, either equation (3) or (4) is applied to the equation of a circle with radius a and it is shown that $\rho = a$ as expected. In this way, the definition for radius of curvature is shown to be reasonable.

There are a number of reasons why the traditional approach is not entirely satisfactory. The definition (equation (2)) for radius of curvature appears to be "pulled out of a hat" as it is not usually part of one's network of knowledge prior to the derivation and so the derivation itself is not well motivated. Since radius of curvature is independent of the coordinate system, it is undesirable to make reference to a coordinate system but this is done in the definition (equation (2)). Although reference is made to a coordinate system in the definition, in the traditional approach it is not shown that the radius of curvature at a point along the curve is independent of rotations or translations of the coordinate system or the type of coordinate system used. The geometrical interpretation of radius of curvature is not emphasized and so one may fail to get an intuitive feeling for the concept. Finally, the elementary traditional treatment does not allow one to actually calculate the radius of curvature for curves which are not confined to a plane.

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[29] T. M. Apostol, *Mathematical Analysis*, Addison Wesley (1957).

[30] T. M. Apostol, *Calculus*, Blaisdell Publishing House (1962).

Here an alternative treatment of radius of curvature is given which overcomes the[illegible] defects. Furthermore, a number of new results are given which do not appear in any [illegible] the consulted references. All of the results for radius of curvature developed in this rep[illegible] are summarized in the appendix. Formulae which do not appear in any of the consult[illegible] references, in the appendix, are designated with an asterisk after the equation numb[illegible]

Almost everyone who has studied elementary plane geometry knows how to fi[illegible] the center of a circle using a straight edge and compass: erect perpendicular bisectors [illegible] two arbitrary chords; the center of the circle is where the perpendicular bisectors m[illegible] This suggests that the radius of curvature [illegible] to a curve $y = f(x)$ at the point x could [illegible] found by: drawing one chord between the points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$ a[illegible] another chord between the points $(x - \Delta x, f(x - \Delta x))$ and $(x, f(x))$; erect perpendicul[illegible] bisectors to these chords and determine as $\Delta x \rightarrow 0$ the coordinates (X, Y) where t[illegible] perpendicular bisectors meet; in this report the radius of curvature [illegible] is defined[34] as t[illegible] distance between (x, y) and (X, Y) (Figure 1). Although it is already known[22] that the cent[illegible] of curvature of a point P on a curve is the limiting position of the normal to the curve at [illegible] with a neighboring normal, a derivation of this result along the lines given here does n[illegible] appear to have been previously done.

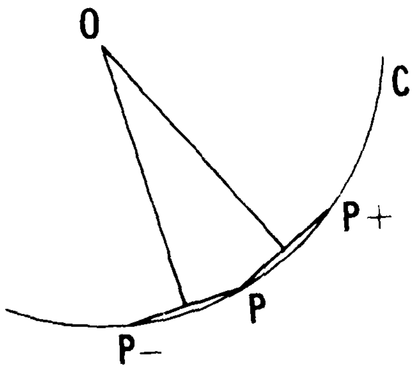

Figure 1. Definition for radius of curvature used in this report

P_-, P and P_+ are three adjacent points on a curve C. The distance [illegible]s from P to P_- equals the [illegible]tance from P to P_+. Perpendicular bisectors of these two chords meet at a point O. The radius [illegible] curvature for the curve C at the point P is defined as the distance from O to P in the limit as [illegible] approaches zero.

[22] W. A. Granville, P. F. Smith and W. R. Longley, [illegible] Differential and Integral Calculus, [illegible] Co. (1934)

[34] The definition for [illegible] given here may appear to differ from the definition given in [illegible] since in [illegible] length of the two chords are made equal whereas in the definition given above the projection of the two [illegible] on the x axis [illegible] are taken to be equal. The two definitions are equivalent since in the limit as [illegible] approaches zero both chord lengths are [illegible] and are equal.

A *heuristic derivation of equation (3) based on the definition of radius of* curvature given above is presented in paragraph 2. A parametric representation of equation (3) is derived in paragraph 3. In paragraph 4, it is shown that the parametric form of equation (3) can be represented in terms of vectors. The vector form of equation (3) shows that ρ is a scalar independent of coordinate system rotations, translations or type. In paragraph 5, it is shown, using physical arguments, that the vector representation of ρ is valid for space curves. Paragraph 6 applies the vector equation for radius of curvature to *compute the radius of curvature in polar coordinates both parametrically and for the case* where r is known as a function of θ. In paragraph 7, it is shown that the definition for radius of curvature adopted in this report agrees with the usual definition, and that the vector formula for radius of curvature derived in paragraph 4 agrees with a more complicated vector relationship given by other authors. In paragraph 8, formulae for radius of curvature applicable to space curves expressed in rectangular, cylindrical, and spherical coordinates are derived. In paragraph 9, formulae for the radius of curvature are derived when the curve is defined implicitly. Paragraph 10 discusses the problem of computing the radius of curvature when the curve is known at a discrete point, and in paragraph 11, the results of paragraph 10 are applied to four engineering problems. The appendix to this report summarizes the various formulae and describes when they are applicable.

II. INVESTIGATION

2. Derivation of Radius of Curvature in Rectangular Coordinates. Let P_-, P and P_+ denote the points $(x - \Delta x, f(x - \Delta x))$, $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$. Then m_1 and m_2 which denote the slopes of the chords from P_- to P and from P to P_+ respectively are:

$$m_1 = \frac{f(x) - f(x - \Delta x)}{\Delta x} ;$$

$$m_2 = \frac{f(x + \Delta x) - f(x)}{\Delta x} .$$

The coordinates (x_1, y_1) of the midpoint of the first chord is $\left(x - \frac{\Delta x}{2}, \frac{f(x) + f(x - \Delta x)}{2}\right)$ and the coordinates (x_2, y_2) of the midpoints of the second chord is $\left(x + \frac{\Delta x}{2}, \frac{f(x) + f(x + \Delta x)}{2}\right)$. The equation of the line perpendicular to the first chord has slope $(-1/m_1)$ and so the equation of the perpendicular bisector of the first chord is

$$\frac{y - y_1}{x - x_1} = -\frac{1}{m_1},$$

which can be rewritten in the form

$$x + m_1 y = x_1 + m_1 y_1. \tag{5a}$$

Similarly, the equation of the perpendicular bisector of the second chord is

$$x + m_2 y = x_2 + m_2 y_2. \tag{5b}$$

The point (X,Y) where equations (5a) and (5b) are both satisfied is the center of the circle defined by the three points P_-, P and P_+:

$$X = \frac{m_2 x_1 - m_1 x_2 - m_1 m_2 (y_2 - y_1)}{m_2 - m_1}; \tag{6a}$$

$$Y = \frac{x_2 - x_1 + m_2 y_2 - m_1 y_1}{m_2 - m_1} \tag{6b}$$

As $\Delta x \rightarrow 0$ the coordinates (X,Y) approach a point in the x,y plane which depends on the point $(x, f(x))$ along the curve. To find the point which (X,Y) approaches as $\Delta x \rightarrow 0$, realize that

$$\lim_{\Delta x \rightarrow 0} \frac{m_2 - m_1}{\Delta x} = y''; \tag{7a}$$

$$\lim_{\Delta x \rightarrow 0} \frac{m_2 + m_1}{2} = y'; \tag{7b}$$

ere y' and y'' are the first and second derivatives at the point x of the curve. Equations a) and (7b) are both reasonable: the first asserts that the difference in slopes divided by x approximates the second derivative and in the limit as $\Delta x \rightarrow 0$ this approximation is act from the definition of second derivative: the second asserts that the average of the pe a little to the left and a little to the right of the point of interest approximates the pe at the point of interest. Equations (7) enable m_1 and m_2 to be written correct to first ler in Δx:

$$m_1 = y' - y'' \frac{\Delta x}{2}; \qquad (7a)'$$

$$m_2 = y' + y'' \frac{\Delta x}{2}. \qquad (7b)'$$

ternately, equations (7a)' and (7b)' could have been written down directly from the ylor series expansion. Similarly, the Taylor series estimates for y_1 and y_2 correct to first ler in Δx are:

$$y_1 = y - y' \frac{\Delta x}{2}; \qquad (8a)$$

$$y_2 = y + y' \frac{\Delta x}{2}. \qquad (8b)$$

ing equations (7)' and (8), as $\Delta x \rightarrow 0$ equations (6) can be expressed in the form:

$$X = \frac{y''x - y' - y'^3}{y''}; \qquad (9a)$$

$$Y = \frac{y''y + y'^2 + 1}{y''}. \qquad (9b)$$

Then the radius of curvature ρ is found from

$$\rho = ((X - x)^2 + (Y - y)^2)^{1/2}. \tag{10}$$

Using equations (9), equation (10) reduces to equation (3).

3. Parametric Representation. The curve $y = f(x)$ could be given in parametric form

$$x = x(t); \; y = y(t), \tag{11}$$

and there is a need to compute the radius of curvature for this case. This happens, for example, when a particle trajectory is found from Newton's law of motion. To express equation (3) in parametric form, realize that

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}} ; \tag{12}$$

$$y'' = \frac{dy'}{dx} = \frac{\frac{d}{dt}\, y'}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)}{\dot{x}} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3} . \tag{13}$$

Substituting equations (11) and (12) into equation (3) yields:

$$\rho = \frac{\left(1 + \frac{\dot{y}^2}{\dot{x}^2}\right)^{3/2}}{\left|\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3}\right|} = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{(\ddot{y}\dot{x} - \dot{y}\ddot{x})} . \tag{14}$$

4. Vector Representation. From the definition of radius of curvature given in this report, it is apparent that the radius of curvature is a property of the curve alone and not of the coordinate system. Thus, the radius of curvature is a scalar and one would expect it to be representable in terms of some scalar combination of vectors. If a radius vector **r** is defined by

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}, \tag{15}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors along the x and y axis, then the velocity vector v is given by:

$$\mathbf{v}(t) = \dot{\mathbf{r}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}}, \tag{16}$$

and the acceleration vector **a** is given by

$$\mathbf{a}(t) = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}}. \tag{17}$$

The numerator of (14) looks like the cube of the magnitude of the velocity vector while the denominator of (14) looks like the z component of the cross product of the acceleration and velocity vectors. With the help of equations (16) and (17) it can be verified that equation (14) can be written in the form

$$\rho = \frac{v^3(t)}{|\mathbf{a}(t) \times \mathbf{v}(t)|} \tag{18}$$

In equation (18) the numerator and denominator are both scalars. Thus, equation (18) shows explicitly that ρ is a scalar independent of coordinate system translations, rotations, or type of coordinate system used.

5. Generalization to Space Curves. Note that equation (18) gives the radius of curvature of a curve confined to the (x,y) plane in terms of the magnitude of the z component of a vector. Suppose now that the curve is no longer confined to the (x,y) plane. In that case, the generalization of equation (15) is

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}. \tag{15'}$$

Realize that the radius of curvature is defined as the limiting position of three points and that three points determine a plane. Thus, in a coordinate system suitably rotated so that the x and y axes are in the plane of the curve, equation (18) would still be valid. But, there is no need to go to a rotated coordinate system because the acceleration and velocity vectors are the same physical vectors in both the original and rotated coordinate systems. Thus, equation (18) is valid for space curves.

6. Representation in Polar Coordinates. In polar coordinates, the radius vector $\mathbf{r}$ is given by

$$\mathbf{r} = r\,\hat{\mathbf{a}}_r \tag{19}$$

where $\hat{\mathbf{a}}_r$ is a unit vector in the direction of $\mathbf{r}$. All three of the quantities in equation (19) can be thought of as functions of time. Differentiating equation (19) with respect to time the velocity is found:[30, 35]

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{a}}_r + r\dot{\theta}\,\hat{\mathbf{a}}_\theta. \tag{20}$$

In equation (20) $\hat{\mathbf{a}}_\theta$ is a unit vector in the direction of increasing θ. The acceleration vector $\mathbf{a}$ is[30, 35]

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{a}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\mathbf{a}}_\theta. \tag{21}$$

From equation (20)

$$v^3(t) = (\dot{r}^2 + r^2\dot{\theta}^2)^{3/2}.$$

From equations (21) and (20)

$$\mathbf{a} \times \mathbf{v} = ((\ddot{r} - r\dot{\theta}^2)r\dot{\theta} - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\dot{r})\,\hat{k}$$

Thus, from equation (18)

$$\rho = \frac{(\dot{r}^2 + r^2\dot{\theta}^2)^{3/2}}{|(\ddot{r} - r\dot{\theta}^2)r\dot{\theta} - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\dot{r}|}. \tag{22}$$

Equation (22) is the polar coordinate analog of equation (14) and is suitable if r and θ are functions of time. To show the reasonableness of equation (22), which is not given in any of the cited references, it can be applied to a general parametric representation of a circle. Such a representation is:

$$r = a; \qquad \theta = \theta(t)$$

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[30] T. M. Apostol, *Calculus*, Blaisdell Publishing House (1962).

[35] K. R. Symon, *Mechanics*, Addison-Wesley (1953).

Substituting the parametric representation of a circle into equation (22) yields $\rho = a$ as expected.

Now, suppose that instead of knowing r and θ as functions of time, that r is known as a function of θ:

$$r = r(\theta). \tag{23}$$

In that case, equation (23) can be parameterized conveniently by the relationship

$$\theta = t; \tag{24a}$$

$$r = r(\theta) = r(t). \tag{24b}$$

Equations (24a) and (24b) are completely equivalent to equation (23). In equation (24), t should be thought of as a general parameter rather than as a time. Although in deriving equation (22), t was thought of as a time, if one reviews the derivation of equation (22) it is apparent that it is in fact applicable to any parameterization of r and θ. From equations (24):

$$\begin{aligned} \dot{r} &= r' & \ddot{r} &= r'' \\ \dot{\theta} &= 1 & \ddot{\theta} &= 0. \end{aligned} \tag{25}$$

In equation (25) dots refer to differentiation with respect to time and primes refer to differentiation with respect to θ. Using equation (25), equation (22) becomes

$$\rho = \frac{(r'^2 + r^2)^{3/2}}{|(r'' - r)r - 2r'^2|} \tag{26}$$

which is an alternate derivation of equation (4).

A similar technique can be used to derive the applicable formula for the case where $\theta = \theta(r)$. The result is given in the appendix as equation (A6).

7. Relationship with Alternate Treatments. The observation that equations (3) and (4) can be derived from both the geometrical interpretation of radius of curvature (paragraph 1) and the conventional definition of radius of curvature (equations (1) and (2)) shows that the two definitions are equivalent.

E. Kreyszig has given the result[27]

$$\rho = \frac{(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{3/2}}{\left((\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})(\ddot{\mathbf{r}}\cdot\ddot{\mathbf{r}}) - (\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})^2\right)^{1/2}} \tag{27}$$

and the same result has been given by Buck[31] in a different notation. To show the equivalence of equation (27) with the result of equation (18) given here, rewrite equation (18) in the form

$$\rho = \frac{(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{3/2}}{\left((\ddot{\mathbf{r}}\times\dot{\mathbf{r}})\cdot(\ddot{\mathbf{r}}\times\dot{\mathbf{r}})\right)^{1/2}}\,. \tag{18$'$}$$

Use the vector identity

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

to write

$$(\ddot{\mathbf{r}}\times\dot{\mathbf{r}})\cdot(\ddot{\mathbf{r}}\times\dot{\mathbf{r}}) = \ddot{\mathbf{r}}\cdot(\dot{\mathbf{r}}\times(\ddot{\mathbf{r}}\times\dot{\mathbf{r}})). \tag{28}$$

Use the vector identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot\mathbf{C}\,\mathbf{B} - \mathbf{A}\cdot\mathbf{B}\,\mathbf{C}$$

to write

$$\dot{\mathbf{r}}\times(\ddot{\mathbf{r}}\times\dot{\mathbf{r}}) = \dot{\mathbf{r}}\cdot\dot{\mathbf{r}}\,\ddot{\mathbf{r}} - \dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}\,\dot{\mathbf{r}}.$$

Thus, equation (28) can be written in the form

$$(\ddot{\mathbf{r}}\times\dot{\mathbf{r}})\cdot(\ddot{\mathbf{r}}\times\dot{\mathbf{r}}) = (\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})(\ddot{\mathbf{r}}\cdot\ddot{\mathbf{r}}) - (\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})^2. \tag{28$'$}$$

and using equation (28)$'$, equation (18)$'$ is shown to be equivalent to (27).

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[27] E. Kreyszig, *Advanced Engineering Mathematics*, John Wiley and Sons, Inc. (1972).

[31] R. C. Buck and E. F. Buck, *Advanced Calculus*, McGraw-Hill, Inc. (1965).

8. Formulae for the Radius of Curvature in Three Dimensions. A generalization of equation (14) valid for a space curve expressed in rectangular coordinates can be found as follows. Differentiate equation (15)′ twice to obtain the velocity and acceleration vectors. Then, the radius of curvature is found from (18):

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{3/2}}{((\dot{y}\ddot{z} - \dot{z}\ddot{y})^2 + (\dot{z}\ddot{x} - \dot{x}\ddot{z})^2 + (\dot{x}\ddot{y} - \dot{y}\ddot{x})^2)^{1/2}} . \qquad (14)'$$

Similarly, a generalization of equation (22) valid for cylindrical coordinates may be obtained. Realize that the generalizations of equation (20) and equation (21) valid for cylindrical coordinates are:[15]

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{a}}_r + r\dot{\theta}\,\hat{\mathbf{a}}_\theta + \dot{z}\,\hat{\mathbf{k}}; \qquad (20)'$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{a}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\mathbf{a}}_\theta + \ddot{z}\,\hat{\mathbf{k}}. \qquad (21)'$$

Then, the parametric formula for the radius of curvature in cylindrical coordinates can be written from equation (18). The result is given in the appendix as equation (A12).

Similarly, a generalization of equation (22) valid for spherical coordinates may be obtained. It is known in spherical coordinates that:[15]

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{a}}_r + r\dot{\theta}\,\hat{\mathbf{a}}_\theta + r\dot{\phi}\sin\theta\,\hat{\mathbf{a}}_\phi;$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\,\hat{\mathbf{a}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,\hat{\mathbf{a}}_\theta$$
$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\,\hat{\mathbf{a}}_\phi.$$

Then, the parametric formula for the radius of curvature in spherical coordinates can be written from equation (18). The result is given in the appendix as equation (A16).

The representation (14)′ is useful when x, y, and z are known as functions of time. Alternately, the equations of the curve might be given in the form:

$$y = y(x); \qquad z = z(x). \qquad (29)$$

To get an explicit formula valid in this case, express equation (29) parametrically by the equations:

$$x = t; \qquad y = y(t); \qquad z = z(t).$$

[15] K. R. Symon, *Mechanics*, Addison-Wesley (1953).

Then, the appropriate formula for the radius of curvature can be found from (14)′. The results are given in the appendix as equation (A9). The same technique is used to derive equations (A10) and (A11) from (A8).

Equation (A12) is useful when the coordinates r, θ, and z are known as functions of t. Equations (A13), (A14), and (A15) which are useful if two of the coordinates are known as a function of the third coordinate are derived by setting the independent variable equal to t and then applying equation (A12).

Similarly, equation (A16) is useful when the coordinates r, θ, and ϕ are known as functions of t. Equations (A17), (A18), and (A19), which are useful if two of the coordinates are known as a function of the third coordinate, are derived by setting the independent variable equal to t and then applying equation (A16).

9. Radius of Curvature for Curves Known Implicitly. The formulae derived up to this point and summarized in the appendix (A1) through (A19) are adequate if the equations of the curve are known parametrically or if the equations of the curve can be explicitly solved for one of the variables. Formulae are needed for the radius of curvature when the equation of the curve is known implicitly.

Consider the curve defined by $F(x,y) = 0$. To find a formula for the radius of curvature realize that the total differential dF is given by

$$dF = F_x\,dx + F_y\,dy = 0, \tag{30}$$

where $F_x \equiv \frac{\partial F}{\partial x}$ and $F_y \equiv \frac{\partial F}{\partial y}$. The derivative $\frac{dy}{dx}$ is readily computed from equation (30):

$$\frac{dy}{dx} = -\frac{F_x}{F_y}\,. \tag{30′}$$

Equation (30)′ gives $\frac{dy}{dx}$ as a function of x and y. By differentiating equation (30)′ again $\frac{d^2y}{dx^2}$ can be computed as a function of x and y:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{F_x}{F_y}\right) = -\frac{1}{dx}\left(\frac{F_y\,dF_x - F_x\,dF_y}{F_y^2}\right)$$

$$= -\frac{1}{dx}\left(\frac{F_y(F_{xx}\,dx + F_{xy}\,dy) - F_x(F_{xy}\,dx + F_{yy}\,dy)}{F_y^2}\right)$$

$$= -\frac{F_yF_{xx} + F_yF_{xy}y' - F_xF_{xy} - F_xF_{yy}y'}{F_y^2} \tag{31}$$

Since y' and y'' are known from equations (30)′ and (31), the radius of curvature can be obtained from equation (A2)

$$\rho = \frac{(F_x^2 + F_y^2)^{3/2}}{|F_y^2F_{xx} - 2F_xF_yF_{xy} + F_x^2F_{yy}|} \tag{32}$$

An alternative derivation of equation (32) can be obtained from equation (A3) by computing x' and x''.

Now, suppose that the radius of curvature is to be computed and the curve is given implicitly in polar coordinates by $F(r,\theta) = 0$. In that case, an argument similar to the one given above shows that

$$\frac{dr}{d\theta} = -\frac{F_\theta}{F_r} \tag{33}$$

and

$$\frac{d^2r}{d\theta^2} = -\frac{(F_\theta^2F_{rr} - 2F_rF_\theta F_{r\theta} + F_r^2F_{\theta\theta})}{F_r^3} \tag{34}$$

Since r' and r'' are known from equations (33) and (34), the radius of curvature can be found from equation (A5):

$$\rho = \frac{(F_\theta^2 + r^2 F_r^2)^{3/2}}{r^2 F_r^3 + r(F_r^2 F_{\theta\theta} - 2F_r F_\theta F_{r\theta} + F_\theta^2 F_{rr}) + 2F_r F_\theta^2} . \tag{35}$$

An alternate derivation of equation (35) can be obtained from equation (A6) by computing θ' and θ''.

To compute the radius of curvature for a space curve given implicitly in cartesian coordinates as the intersection of the surface $F(x,y,z) = 0$ with the surface $G(x,y,z) = 0$ proceed as follows. First, from the total differential of the functions F and G:

$$F_x\,dx + F_y\,dy + F_z\,dz = 0; \tag{36a}$$

$$G_x\,dx + G_y\,dy + G_z\,dz = 0. \tag{36b}$$

Now, suppose x is considered the independent variable so that the appropriate equation to use is equation (A9). Alternately, y and z could be considered the independent variable and equations (A10) or (A11) used. From equation (36), the derivatives $\frac{dy}{dx}$ and $\frac{dz}{dx}$ are readily computed by Cramer's rule:

$$\frac{dy}{dx} = \frac{-\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} , \qquad \frac{dz}{dx} = \frac{-\begin{vmatrix} F_y & F_x \\ G_y & G_x \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} . \tag{37}$$

All of the quantities which appear in equation (37) are functions of x, y, and z. Thus, with the help of equation (37), $\frac{d^2y}{dx^2}$ and $\frac{d^2z}{dx^2}$ can be computed as a function of the points x, y, and z along the curve and then the radius of curvature can be computed from equation (A9). An explicit formula is not exhibited for this case.

A similar procedure can be used if the equation of the curve is given implicitly in cylindrical coordinates as the intersection of the surface $F(r,\theta,z) = 0$ with the surface $G(r,\theta,z)=0$. If θ is regarded as the independent variable then $\frac{dr}{d\theta}$ and $\frac{dz}{d\theta}$ can be computed:

$$\frac{dr}{d\theta} = \frac{-\begin{vmatrix} F_\theta & F_z \\ G_\theta & G_z \end{vmatrix}}{\begin{vmatrix} F_r & F_z \\ G_r & G_z \end{vmatrix}} ; \qquad \frac{dz}{d\theta} = \frac{-\begin{vmatrix} F_r & F_\theta \\ G_r & G_\theta \end{vmatrix}}{\begin{vmatrix} F_r & F_z \\ G_r & G_z \end{vmatrix}} . \tag{38}$$

From equation (38) r'' and z'' can be computed and then the radius of curvature found from equation (A14).

The same technique described above for rectangular and cylindrical coordinates can be used to compute the radius of curvature in spherical coordinates when the curve is known implicitly.

10. Radius of Curvature for Curves Known at Discrete Points. The formulae derived or described in the preceeding section can be used when the equation of the curve is known in an analytical form which can be differentiated at least twice. In practical circumstances, the curve is often known only at discrete points and there is a need to compute the radius of curvature for this case. Several approaches can be used.

One approach is to use the value of the function at the discrete points to evaluate the appropriate first and second derivatives. Formulae for doing this are given in numerical analysis books[36-38] and research articles.[39] Then the appropriate formula choosen from equations (A1) through (A24) can be used to evaluate the radius of curvature.

[36] M. G. Salvadori and M. L. Baron, *Numerical Methods in Engineering*, Prentice Hall, Inc. (1964).

[37] L. Kelly, *Handbook of Numerical Methods and Applications*, Addison Wesley Publishing Co. (1967).

[38] F. Scheid, *Numerical Analysis*, McGraw Hill Book Co. (1968).

[39] A. Savitzky and M. J. E. Golay, "Smoothing and Differentiation of Data by Simplified Least Squares Procedures," Anal. Chem. *36*, 1627 (1964).

Another approach is to fit the discrete points with a continuous, twice differentiable function that approximates the discrete points. The fit might be done by the method of least squares[32,37,39] or by any other method judged appropriate. From the fitted analytical formula, the first and second derivatives can be computed and then the radius of curvature found from the appropriate equation (A1) through (A24).

In this paragraph, a method is developed for computing the radius of curvature when the curve is known at three adjacent discrete points. The idea is based on the following observations. Any three points in space define a plane providing no two points coincide and the three points are not co-linear. These three points determine a circle in that plane and the radius of curvature along the curve between the three points is here defined to be the radius of that circle.

To calculate the radius of curvature, denote the coordinates of the center point and the two adjacent points relative to an arbitrary origin by $\mathbf{r}_0$, $\mathbf{r}_1$, and $\mathbf{r}_2$ respectively. Denote the vector from $\mathbf{r}_0$ to $\mathbf{r}_1$ by $\mathbf{v}_1$ and the vector from $\mathbf{r}_0$ to $\mathbf{r}_2$ by $\mathbf{v}_2$. Let $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ denote unit vectors in the direction of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then,

$$\mathbf{v}_1 = \mathbf{r}_1 - \mathbf{r}_0; \qquad \mathbf{v}_2 = \mathbf{r}_2 - \mathbf{r}_0; \tag{39a}$$

$$\hat{\mathbf{n}}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}; \qquad \hat{\mathbf{n}}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|} \tag{39b}$$

Denote a unit vector normal to the plane of $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ by $\hat{\mathbf{n}}_3$. Then,

$$\hat{\mathbf{n}}_3 = \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2. \tag{40}$$

A unit vector normal to $\mathbf{v}_1$ and in the plane of $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\hat{\mathbf{n}}_3 \times \hat{\mathbf{n}}_1$. Similarly, a unit vector normal to $\mathbf{v}_2$ and in the plane of $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\hat{\mathbf{n}}_3 \times \hat{\mathbf{n}}_2$. Thus, the parametric equation of a line perpendicular to $\mathbf{v}_1$ at the midpoint of $\mathbf{v}_1$ and in the plane of $\mathbf{v}_1$ and $\mathbf{v}_2$ is

[32] K. Rektorys, editor, *Survey of Applicable Mathematics*, The MIT Press (1969).

[37] L. Kelly, *Handbook of Numerical Methods and Applications*, Addison-Wesley Publishing Co. (1967).

[39] A. Savitzky and M. J. E. Golay, "Smoothing and Differentiation of Data by Simplified Least Squares Procedures," Anal. Chem. *36*, 1627 (1964).

$$\mathbf{r} = \frac{v_1}{2} + (\hat{\mathbf{n}}_3 \times \hat{\mathbf{n}}_1)t = \frac{v_1}{2}\hat{\mathbf{n}}_1 + t((\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \times \hat{\mathbf{n}}_1), \tag{11}$$

where the origin associated with the vector $\mathbf{r}$ is the point $\mathbf{r}_0$. The triple vector cross product in (11) can be reduced

$$(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \times \hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2 - \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2\, \hat{\mathbf{n}}_1,$$

so that equation (11) can be written in the form

$$\mathbf{r} = \left(\frac{v_1}{2} - t\,\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2\right)\hat{\mathbf{n}}_1 + t\hat{\mathbf{n}}_2 \tag{11'}$$

Similarly, the parametric equation of a line perpendicular to $\mathbf{v}_2$ at the midpoint of $\mathbf{v}_2$ and in the plane of $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$\mathbf{r} = t\hat{\mathbf{n}}_1 + \left(\frac{v_2}{2} - t\,\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2\right)\hat{\mathbf{n}}_2 \tag{12}$$

The intersection of the locus of points defined by (11') with the locus of points defined by (12) define the center of the circle. To find this point let r_1 and r_2 denote the components of $\mathbf{r}$ in the $\mathbf{n}_1$ and $\mathbf{n}_2$ directions. Then,

$$\mathbf{r} = r_1\hat{\mathbf{n}}_1 + r_2\hat{\mathbf{n}}_2 \tag{13}$$

Comparison of (13) and (11') implies

$$r_1 = \frac{v_1}{2} - t\,\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2;$$

$$r_2 = t.$$

and this in turn implies equation (41) can be expressed non-parametrically by

$$r_1 + \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 \, r_2 = \frac{v_1}{2} \tag{44a}$$

Similarly, equation (42) is expressed non-parametrically by

$$\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 \, r_1 + r_2 = \frac{v_2}{2} \tag{44b}$$

The center of the circle is where equations (44a) and (44b) are simultaneously satisfied. This is found from (44) by Cramer's rule:

$$r_1 = \frac{v_1 - \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 \, v_2}{1 - (\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2)^2}, \tag{45a}$$

$$r_2 = \frac{v_2 - \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 \, v_1}{1 - (\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2)^2}. \tag{45b}$$

Equation (45) is valid for non-colinear points; i.e., $\hat{\mathbf{n}}_1$ cannot be parallel or antiparallel to $\mathbf{n}_2$ or the expression (45) becomes indeterminate. The radius of curvature ρ is the distance *from $\mathbf{r}_0$ to the center of the circle:*

$$\begin{aligned} \rho &= (\mathbf{r} \cdot \mathbf{r})^{1/2} = ((r_1 \hat{\mathbf{n}}_1 + r_2 \hat{\mathbf{n}}_2) \cdot (r_1 \hat{\mathbf{n}}_1 + r_2 \hat{\mathbf{n}}_2))^{1/2} \\ &= (r^2_1 + 2 r_1 r_2 \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 + r^2_2)^{1/2} \end{aligned} \tag{46}$$

Equations (39), (45), and (46) allow the radius of curvature to be computed numerically in any coordinate system and is the final result of this section. These equations are a generalization of a well-known geometrical theorem. In Figure 2, AC is a diameter of a semi-circle with P an arbitrary point on the semi-circle. Denote by $\mathbf{v}_1$ and $\mathbf{v}_2$ the vectors PA and PC. The well-known geometrical result is that $\mathbf{v}_1$ is normal to $\mathbf{v}_2$ and so from Figure 2, the radius ρ of the circle is:

$$\rho = \frac{(|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2)^{1/2}}{2}$$

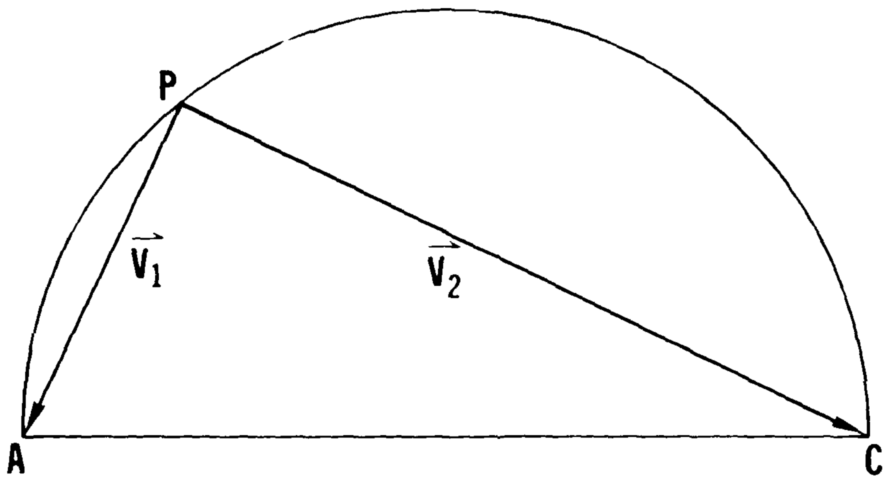


Figure 2. Generalization of a well-known geometrical result.

AC is a diameter of a circle with arc APC. P is an arbitrary point on the circumference of the circle and if v_1 and v_2 go from P to A and from P to C then a well-known geometrical result asserts that the vectors v_1 and v_2 are orthogonal and this enables the radius of the circle to be readily computed in terms of $|v_1|$ and $|v_2|$. In paragraph 10, a formula for the radius of the circle is found when v_1 and v_2 are vectors from P to arbitrary points on the circumference of the circle, and in section 12 the results of paragraph 10 are specialized to the case where $|v_1| = |v_2|$.

This result is obtained directly from equations (45) and (46) for the special case $\hat{n}_1 \cdot \hat{n}_2 = 0$.

III. APPLICATIONS

11. Aircraft Maneuverability. The maneuverability of enemy aircraft is of considerable interest to the friendly fighter aircraft designer. One parameter which measures the maneuverability of an aircraft is the path length necessary for it to change the direction of its center of mass through a given angle. This quantity is measured by the smallest attainable radius of curvature for the curve defined by the center of mass along a flight path and can be computed in the following way. Suppose from radar data that the spherical coordinates r_i, θ_i, ϕ_i of an enemy aircraft are known at the ith time instant. The spherical coordinates at the i, i−1, and i+1 time instants define radius vectors $\mathbf{r}_i$, $\mathbf{r}_{i-1}$, and $\mathbf{r}_{i+1}$. Then, using equations (39), (45), and (46) the radius of curvature associated with the ith point can be computed. This can be done for all i and the radius of curvature exhibited for all points along the flight path.

The success of the scheme described above depends on the accuracy of the radar and the frequency with which the data is taken. The one standard deviation error associated with the ith radar measurement can be represented as a sphere of radius R_i centered about the tip of the vector $\mathbf{r}_i$ by which we mean that about 68 percent of the time the true value of the radius vector will be within a distance R_i of the measured value of the radius vector. Denote the distance between $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ by $\Delta \mathbf{r}_i$. Then, necessary conditions for the radar to accurately determine the minimum radius of curvature ρ_{min} are:

$$\rho_{min} >> \max(R_i), \qquad \rho_{min} \gtrsim |\Delta \mathbf{r}_i|.$$

12. Curvature of Railroad Track. One parameter of interest to engineers responsible for maintaining railroads is the horizontal curvature of a railroad track. Here a method for measuring this quantity is described. In Figure 3 let AD represent a section of track. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are drawn from an arbitrary point P along the track in opposite directions to points B and C along the track. For simplicity, the length of $\mathbf{v}_1$ is chosen to be equal to the lenght of $\mathbf{v}_2$; i.e.,

$$|\mathbf{v}_1| = |\mathbf{v}_2| . \tag{47}$$

In practice $\mathbf{v}_1$ and $\mathbf{v}_2$ represent rigid rods of equal length mounted on wheels located at B, P, and C. The rods are joined at P so that they can both rotate freely about the point P. In that case the radius of curvature of the track between the end points of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is determined by the length of the rigid rod and the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ as the following argument shows.

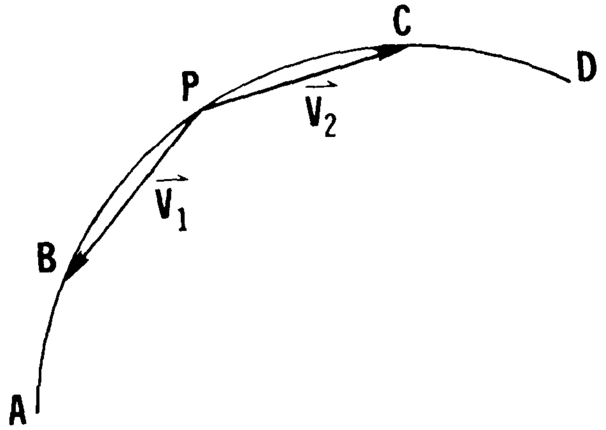


Figure 3. Measuring railroad track horizontal radius of curvature.
AD represents a top view of a railraod track section. Points B, P, and C are point on the track (mounted on wheels) which can move. BP and PC represent rigid rods of equal length $|v_1|$ joined at P so that they can rotate about a vertical axis through P. Then, the horizontal radius of curvature is determined by $|v_1|$ and the angle between the vectors v_1 and v_2.

The explicit formula for the radius of curvature when $|v_1| = |v_2|$ is a special case of the equations (43), (45), and (46) of paragraph 10. Equations (43) and (45) imply

$$r = \frac{|v_1| - \hat{n}_1 \cdot \hat{n}_2 \, |v_2|}{(1 - (\hat{n}_1 \cdot \hat{n}_2)^2)} \frac{v_1}{|v_1|} + \frac{|v_2| - \hat{n}_1 \cdot \hat{n}_2 \, |v_1|}{(1 - (\hat{n}_1 \cdot \hat{n}_2)^2)} \frac{v_2}{|v_2|}$$

$$= \frac{1}{(1 - (\hat{n}_1 \cdot \hat{n}_2)^2)} \left(\left(1 - \hat{n}_1 \cdot \hat{n}_2 \frac{|v_2|}{|v_1|}\right) v_1 + \left(1 - \hat{n}_1 \cdot \hat{n}_2 \frac{|v_1|}{|v_2|} \, v_2\right)\right). \quad (48)$$

Using (47), equation (48) reduces to

$$r = \frac{1}{(1 + \hat{n}_1 \cdot \hat{n}_2)} \left(\frac{v_1 + v_2}{2} \right)$$

and

$$\rho = (\mathbf{r}\cdot\mathbf{r})^{1/2} = \frac{(|\mathbf{v}_1|^2 + 2\mathbf{v}_1\cdot\mathbf{v}_2 + |\mathbf{v}_2|^2)^{1/2}}{2(1 + \hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2)} . \tag{49}$$

Using (17), equation (49) reduces to

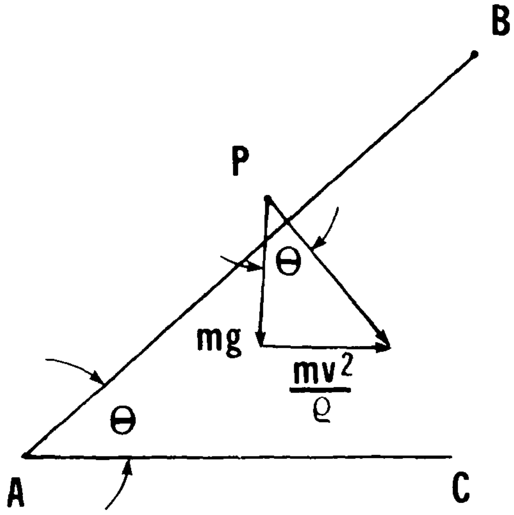
$$\rho = \frac{(|\mathbf{v}_1|^2 + \mathbf{v}_1\cdot\mathbf{v}_2)}{\sqrt{2}\,(1 + \hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2)} = \frac{|\mathbf{v}_1|}{\sqrt{2}\,(1 + \hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2)} \tag{50}$$

and this result can be used to measure the horizontal radius of curvature of the track for the device described above.

Note equation (50) asserts that if $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are antiparallel, which corresponds to a straight track, then ρ is infinite as expected. Note also that equation (50) asserts that if $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are parallel, then $\rho = \frac{|\mathbf{v}_1|}{2}$. This conflicts with intuition since such a curve corresponds to a very sharp curve and one would expect ρ to be zero. Note that if the length of $|\mathbf{v}_1|$ is made small enough, then ρ can be made to approach zero for this case. From this we learn that in using (50) the length $|\mathbf{v}_1|$ should be no larger than twice the smallest radius of curvature that the device is expected to measure.

With a suitably choosen $|\mathbf{v}_1|$ and vertically hinged rigid rods, equation (50) could also be used for measuring the vertical radius of curvature of railroad track.

13. Banking of Roads. High speed roads are often banked so that an automobile going at the recommended speed will feel an effective force perpendicular to the road surface. This reduces the possibility of the automobile slipping along the road toward the outside of the curve because of centrifugal force when the surface friction is reduced because of rain or snow. As Figure 4 shows, an automobile travelling with speed v along a road with radius of curvature ρ and banked at an angle θ will have an effective force perpendicular to the road, provided the speed v satisfies the relationship $v = (\rho g \tan\theta)^{1/2}$. As a third application of the work done in this report, a device is described which can determine if the radius of curvature and the angle of bank are properly adjusted for the speed v at each point along the curve.



$$\tan \Theta = \frac{\frac{mv^2}{\varrho}}{mg} = \frac{v^2}{\varrho g}$$

Figure 4. The proper banking for a level curved road.

P represents the center of mass of an automobile moving with speed v perpendicular to the plane of the paper. AB represents a road surface banked at an angle θ with respect to the horizontal AC. The road is curved with radius of curvature ρ. The net effective force acting on the automobile has two components: a vertical component mg and a horizontal component mv^2/ρ. The diagram shows that when the condition $\tan \theta = v^2/(\rho g)$ is satisfied that the effective force acting on the automobile is perpendicular to the road surface.

A device for accomplishing this is sketched in Figure 5. AB, BC, and BD all represent rigid rods mounted on wheels and each of the points A, B, C, and D are a conveniently chosen height h above the ground. For convenience, rods BA and BC are chosen to be of equal length with rod BD and, also, conveniently chosen to be equal to a lane width or some integer multiple of a lane width. Rods AB, BC, and BD have holes drilled so that they can all rotate freely about a vertical pin located at point B. Four people located at points A, B, C, and D are required to operate the device. As people located at points A, B, and C walk along the road surface, each one keeps the wheels on the painted line which defines the outside of the curve while the person at D keeps the wheel on the painted line which defines the inside of the curve and also keeps the rod BD approximately perpendicular to both curves. Furthermore, all four people keep the rods of length h approximately vertical. The device has two angle measuring instruments at B. One measures the angle between the rods BA and BC. The cosine of this angle equals $n_1 \cdot n_2$ in equation (50). The other instrument measures the angle between BD and the horizontal and corresponds to the angle θ defined in Figure 4. A small programmable computer takes as inputs the length of the rod AB, the angle ABC, and the angle θ. Then the radius of curvature ρ is computed from equation (50) and the optimum speed from the relationship $v = (\rho g \tan \theta)^{1/2}$. If the two angles are fed automatically and continuously into a programmable calculator as the device is moved along the curve, than an operator at B can continuously monitor the optimum speed computed by the programmable calculator and check that it is constant within certain limits.

The idea illustrated in Figure 5 is also applicable to trains and railroad track.

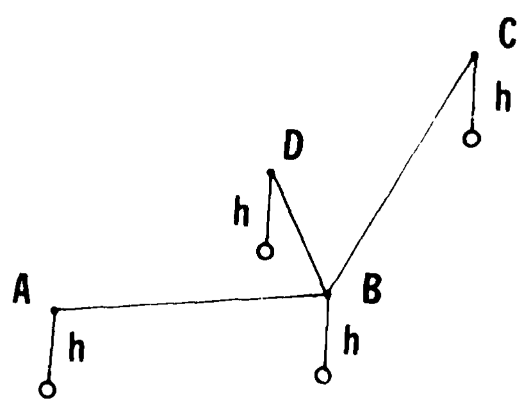


Figure 5. Device for measuring optimum automobile speed on a banked, curved road. The device can determine the speed of an automobile such that the effective force acting on it will be perpendicular to the road surface.

11. Vehicular Land Navigation System. The radius of curvature idea can be used as a basis for a system, here called a vehicular land navigation system (Figure 6), where a moving vehicle will be capable of automatically recording its current x and y coordinates relative to an arbitrarily placed coordinate system. If such a system could be made to work reliably it would have at least two military uses.

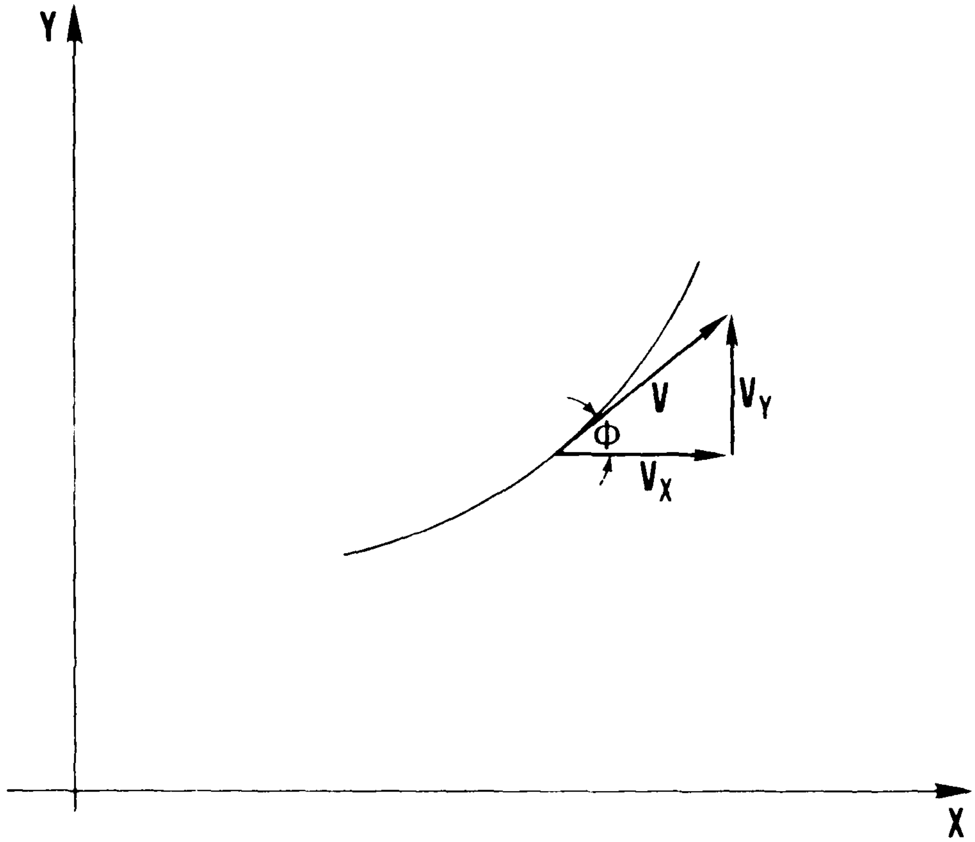


Figure 6. Principle of navigation system using radius of curvature.

By knowing the radius of curvature $\rho(t)$ and the speed $v(t)$ for all time and the initial angle ϕ_0, the angle ϕ is known for all time (see paragraph 11). Knowing $v(t)$ and $\phi(t)$ the x and y components of velocity are known for all time as the figure above shows. By numerically integrating $v_x(t)$ and $v_y(t)$ the position of the vehicle is known for all time.

A vehicle equipped with this equipment would be useful for reconnaissance. When the operator of the vehicle encountered objects of interest (for example, mine fields) a coded button could be pushed and the coordinates and a code identifying the item of interest would be recorded on a magnetic card or stored in the memory of a programmable calculator. At the base camp or in the vehicle the information could be recalled and recorded on a map under less hazardous conditions.

The second application involves control of resources by a commander. The automatic vehicular navigation system would continuously update its coordinates and place them in a register connected to a radio transmitter normally turned off. When a commander queried a certain vehicle, the transmitter would automatically respond with its coordinates and with a completely automated system might be able to do this quickly enough so that the enemy would not have time to get a fix. A more intricate system also could quickly and automatically transmit any intelligence stored on the magnetic card described in the first system.

How could a vehicular land navigation system be made? One way to do this would be to have a gyroscope-mounted system which would point in a reference direction. When the vehicle's speed and the direction of the vehicle's motion relative to the reference direction are known, the coordinates of the vehicle could be continuously updated using a programmable calculator. Since gyroscopes are expensive and fragile, possibly a less expensive and more practical way to implement such a system would be to use a conventional compass to point to the reference direction. A difficulty with a conventional compass is that if the vehicle goes over a bump or passes by a magnetized object such as a tank, then the compass will turn despite the fact that the vehicle may not have changed direction and so the compass gives a false reading. If the angle the front wheel makes with the forward axis[36] of the vehicle is known, along with the speed of the vehicle and this information is compared with the angular movements of the compass wheel (possibly measured optically), the fact that the compass is acting erratically can be determined automatically by the system. For example, if the vehicle is moving in a straight line as determined by the speedometer, and the angle of the front wheel with the forward axis of the vehicle and the compass is moving about a broad angular range, then the system would know that the compass is acting erratically. Thus, with a conventional compass, there is a need for updating the vehicle's coordinates during the time when the compass is known to be inaccurate. This need can be satisfied by a system based on the radius of curvature concept which will now be described.

[36] The direction of the vehicle's forward axis is defined as the direction the vehicle would travel on a plane level surface with the steering wheel dead center. The steering wheel is at dead center when the curve defined by the path of the vehicle has an infinite radius of curvature.

In the discussion up to this point, the radius of curvature has been defined in such a way that it is always positive. A vehicle moving in a straight line follows a path with zero curvature and one would expect that positive or negative curvature could be assigned to the vehicle's path depending on whether the steering wheel is turned to the left or the right. As Figure 7 shows, a vehicle moving counterclockwise corresponds to s and ϕ increasing at the same time and so if equation (1) is written *without an absolute value* sign, this leads to positive curvature in equation (3). Thus, in a vehicle moving forward, positive radius of curvature *corresponds to turning* the steering wheel to the left from dead center. Similarly, when the steering wheel is to the right of dead center, this *corresponds to clockwise motion* for a forward moving vehicle and a negative radius of curvature. The magnitude of the angle which the front wheels make relative to the forward axis of the vehicle determines the magnitude of the positive or negative radius of curvature for the vehicle. The relationship between these quantities can be measured experimentally by setting the wheel at an arbitrary angle θ with respect to the forward axis of the vehicle and then measuring the turning radius associated with this angle. This information can be stored in the programmable calculator so that when a sensor measures θ the programmable calculator can, in real time, convert this measurement to radius of curvature. We suppose that there are *measurement systems on* the vehicle which, at any instant of time, tell the magnitude of the velocity vector $v(t)$ and the radius of curvature $\rho(t)$. Then,

$$\frac{1}{\rho} = \frac{d\phi}{ds} = \frac{\frac{d\phi}{dt}}{\frac{ds}{dt}} = \frac{\frac{d\phi}{dt}}{v} ,$$

which can be rewritten in the form

$$\frac{d\phi}{dt} = \frac{v(t)}{\rho(t)} , \qquad (51)$$

or

$$\phi(t) = \int_{t=t_0}^{t} \frac{v(t)}{\rho(t)}\, dt + \phi_0 . \qquad (51)'$$

Equation (51)' asserts that if $v(t)$ and $\rho(t)$ are known at each instant of time, and if ϕ_0 is also known initially, then the angle which the tangent or velocity vector makes with the positive x axis will be known for each instant of time. This means that the projections of the velocity vector on the x and y axes are known for all time:

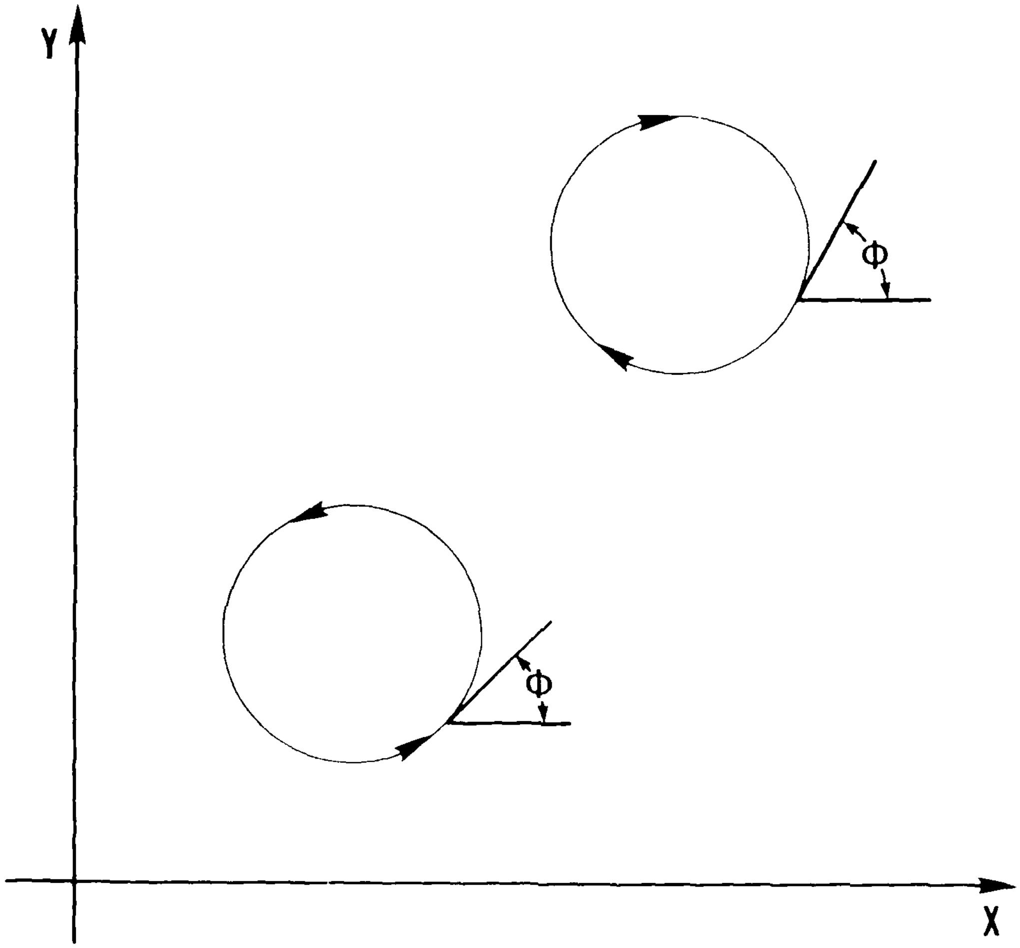


Figure 7. The sign for radius of curvature.

If one imagines a vehicle travelling along a curve, those points where the steering wheel is turned toward the left correspond to positive radius of curvature while those points where the steering wheel is turned toward the right correspond to negative radius of curvature. The reason for this is that in counter-clockwise motion s and ϕ increase together while for clockwise motion increasing s is associated with decreasing ϕ.

$$\frac{dx}{dt} = v(t) \cos \phi(t); \qquad (52b)$$

$$\frac{dy}{dt} = v(t) \sin \phi(t). \qquad (52c)$$

Equations (51)' and (52) are readily integrated numerically by the programmable calculator:

$$\phi_{n+1} = \phi_n + \frac{v_n}{\rho_n} \Delta t, \qquad (53a)$$

$$x_{n+1} = v_n \Delta t \cos \phi_n; \qquad (53b)$$

$$y_{n+1} = v_n \Delta t \sin \phi_n. \qquad (53c)$$

Equation (53) makes quantitative the following intuitive result:

If the initial coordinates and angle ϕ for the vehicle are known, then a knowledge of the vehicle's speed and the angle θ which the front wheels make with the vehicle's forward axis for all subsequent time enables the vehicle position, at all subsequent time, to be predicted. Although equation (53) can conceptually be used to track the vehicle for all time, in practice, the technique is only expected to be accurate for time intervals which are not too long. The reason for this is that uncertainties in v_n and ρ_n in equation (53a) after a long time interval lead to uncertainties in ϕ_{n+1} and so the information telling which way the vehicle is pointing eventually gets lost, at which time further integration of equation (53) is no longer meaningful. The length of time for this to happen depends on the magnitudes of ρ_n, v_n, the precision with which these quantities are measured, and the uncertainties which can be tolerated in the final postion.

IV. CONCLUSIONS

15. Conclusions. The radius of curvature treatment given in paragraph 2, while algebraically more difficult than the usual treatment might be expected to appeal to engineers and scientists because it addresses the problem of computing the radius of curvature in a direct and intuitive way. It provides an example of the interplay between geometry, numerical methods, and calculus in evaluating a limit which some readers may find instructive.

Some techniques suitable for the numerical evaluation of radius of curvature are given. For curves confined to the plane, equations (6) and (10) may be used to find the center and radius of curvature when the functional values but not the derivatives are known. Equations (39), (45), and (46) generalize these results to space curves.

The radius of curvature treatment given in this report makes use of kinematic and vector methods, techniques familiar to engineers and scientists and so this particular treatment may appeal to these people more than the traditional approach.

A search of the literature failed to find the equations designated with an asterisk in the appendix. These and the applications are the new results of this report. Engineers and scientists who have a need to compute the radius of curvature may find this compilation of formulae and the methods used to derive them useful.

In closing, the reader is left with some conjectures dealing with two aspects of this subject.

When the equation of the curve is known implicitly, equations (A2) and (A3) yield the same formula (A20) for the radius of curvature in cartesian coordinates. Also, when the equation of the curve is known implicitly, equations (A5) and (A6) yield the same formula (A21) for the radius of curvature in polar coordinates. It is conjectured that for cartesian space curves known implicitly, equations (A9), (A10) and (A11) all yield the same formula for the radius of curvature, and a similar result will be true in cylindrical and spherical coordinates.

Because the radius of curvature is a scalar independent of coordinate system translations, rotations, or type it was possible to express equation (A1) in vector form and the vector form of the equation (A7) was useful in deriving formulae for radius of curvature in three dimensions. Similarly, it is conjectured that equation (A20) can be represented as a scalar combination of vectors. For example, the numerator of equation (A20) is $(\nabla F \cdot \nabla F)^{3/2}$. But how can the denominator be expressed as a scalar combination of vectors? If equation (A20) could be expressed as a scalar combination of vectors, then equation (A21) might be derived directly and straightforwardly from equation (A20) by using known vector results.

APPENDIX

SUMMARY OF RESULTS

Note: Formulae which do not appear in any of the consulted references of this report are indicated by an asterisk after the equation number.

A-1. Two dimensional curves.

Cartesian coordinates.

1. Equation of curve given parametrically by: $x = x(t)$, $y = y(t)$. Then

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{|\ddot{y}\dot{x} - \dot{y}\ddot{x}|} , \qquad (A1)$$

$$\dot{x} = \frac{dx}{dt} \text{, etc.}$$

2. Equation of curve given by: $y = y(x)$. Then

$$\rho = \frac{(1 + y'^2)^{3/2}}{|y''|} ; \qquad (A2)$$

$$y' = \frac{dy}{dx} \text{, etc.}$$

3. Equation of curve given by: $x = x(y)$. Then

$$\rho = \frac{(x'^2 + 1)^{3/2}}{|x''|} \qquad (A3)$$

$$x' = \frac{dx}{dy} \text{, etc.}$$

Polar coordinates.

1. Equation of curve given parametrically by $r = r(t), \theta = \theta(t)$. Then

$$\rho = \frac{(\dot{r}^2 + r^2\dot{\theta}^2)^{3/2}}{|(\ddot{r} - r\dot{\theta}^2)r\dot{\theta} - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\dot{r}|} . \qquad \text{(A4)*}$$

$\dot{r} = \dfrac{dr}{dt}$, etc.

2. Equation of curve given by $r = r(\theta)$. Then

$$\rho = \frac{(r'^2 + r^2)^{3/2}}{|(r'' - r)r - 2r'^2|} \qquad \text{(A5)}$$

$r' = \dfrac{dr}{d\theta}$, etc.

3. Equation of curve given by $\theta = \theta(r)$. Then

$$\rho = \frac{(1 + r^2\theta'^2)^{3/2}}{r\theta'' + r^2\theta'^3 + 2\theta'} . \qquad \text{(A6)*}$$

$\theta' = \dfrac{d\theta}{dr}$, etc.

A-2. Space curves.

General result. Equation of curve is given parametrically by

$\mathbf{r} = \mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$; $\mathbf{v} = \dot{\mathbf{r}}(t)$; $\mathbf{a} = \ddot{\mathbf{r}}(t)$. Then

$$\rho = \frac{v^3(t)}{|\mathbf{a}(t) \times \mathbf{v}(t)|} \qquad \text{(A7)}$$

Cartesian coordinates.

1. Equation of curve is given parametrically by: $x = x(t)$, $y = y(t)$, $z = z(t)$. Then:

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{3/2}}{((\dot{y}\ddot{z} - \ddot{z}\dot{y})^2 + (\dot{z}\ddot{x} - \ddot{x}\dot{z})^2 + (\dot{x}\ddot{y} - \ddot{y}\dot{x})^2)^{3/2}}; \qquad \text{(A8)'}$$

$\dot{x} = \dfrac{dx}{dt}$, etc.

2. Equation of curve is given by: $y = y(x)$, $z = z(x)$. Then

$$\rho = \frac{(1 + y'^2 + z'^2)^{3/2}}{(y''^2 + z''^2 + (y''z' - z''y')^2)^{1/2}}; \qquad \text{(A9)'}$$

$y' = \dfrac{dy}{dx}$, etc.

3. Equation of curve is given by: $x = x(y)$, $z = z(y)$. Then

$$\rho = \frac{(x'^2 + 1 + z'^2)^{3/2}}{(x''^2 + z''^2 + (z''x' - x''z')^2)^{1/2}}; \qquad \text{(A10)'}$$

$x' = \dfrac{dx}{dy}$, etc.

4. Equation of curve is given by: $x = x(z)$, $y = y(z)$. Then

$$\rho = \frac{(x'^2 + y'^2 + 1)^{3/2}}{(x''^2 + y''^2 + (x''y' - y''x')^2)^{1/2}} \qquad \text{(A11)'}$$

Cylindrical coordinates.

1. Equation of curve given parametrically by: $r = r(t)$, $\theta = \theta(t)$, $z = z(t)$. Then:

(A11)

$$\rho = \frac{(\dot r^2 + r^2\dot\theta^2 + \dot z^2)^{3/2}}{\left(((r\ddot\theta + 2\dot r\dot\theta)\dot z - r\dot\theta\ddot z)^2 + (\dot r\ddot z - (\ddot r - r\dot\theta^2)\dot z)^2 + ((\ddot r - r\dot\theta^2)r\dot\theta - (r\ddot\theta + 2\dot r\dot\theta)\dot r)^2\right)^{1/2}}$$

(A12)

$\dot r = \dfrac{dr}{dt}$, etc.

2. Equation of curve given by: $\theta = \theta(r)$, $z = z(r)$. Then

$$\rho = \frac{(1 + r^2\theta'^2 + z'^2)^{3/2}}{\left(((r\theta'' + 2\theta')z' - r\theta' z'')^2 + (z'' + r\theta'^2 z')^2 + (r^2\theta'^3 + r\theta'' + 2\theta')^2\right)^{1/2}}$$

(A13)

$\theta' = \dfrac{d\theta}{dr}$, etc.

3. Equation of curve given by: $r = r(\theta)$, $z = z(\theta)$. Then

$$\rho = \frac{(r'^2 + r^2 + z'^2)^{3/2}}{\left((2r'z' - rz'')^2 + (r'z'' - (r'' - r)z')^2 + ((r'' - r)r - 2r'^2)^2\right)^{1/2}}$$

(A14)

$r' = \dfrac{dr}{d\theta}$, etc.

4. Equation of curve given by: $r = r(z)$, $\theta = \theta(z)$. Then

$$\rho = \frac{(r'^2 + r^2\theta'^2 + 1)^{3/2}}{\left((r\theta'' + 2r'\theta')^2 + (r'' - r\theta'^2)^2 + ((r'' - r\theta'^2)r\theta' - (r\theta'' + 2r'\theta')r')^2\right)^{1/2}}$$

(A15)

$$r' = \frac{dr}{dz}, \text{ etc.}$$

Spherical coordinates.

1. Equation of curve given parametrically by: $r = r(t)$, $\theta = \theta(t)$, $\phi = \phi(t)$. Then

$$\rho = \frac{(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta)^{3/2}}{((Br\dot{\phi}\sin\theta - Cr\dot{\theta})^2 + (C\dot{r} - Ar\dot{\phi}\sin\theta)^2 + (Ar\dot{\theta} - B\dot{r})^2)^{1/2}}, \qquad \text{(A16)*}$$

where

$$A = \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta$$

$$B = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta$$

$$C = r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta$$

2. Equation of curve given by: $\theta = \theta(r)$, $\phi = \phi(r)$. Then

$$\rho = \frac{(1 + r^2\theta'^2 + r^2\phi'^2 \sin^2\theta)^{3/2}}{((Br\phi'\sin\theta - Cr\theta')^2 + (C - Ar\phi'\sin\theta)^2 + (Ar\theta' - B)^2)^{1/2}}, \qquad \text{(A17)*}$$

where

$$A = -r\theta'^2 - r\phi'^2 \sin^2\theta$$

$$B = r\theta'' + 2\theta' - r\phi'^2 \sin\theta\cos\theta$$

$$C = r\phi'' + 2\phi'\sin\theta + 2r\theta'\phi'\cos\theta$$

3. Equation of curve given by: $r = r(\theta)$, $\phi = \phi(\theta)$. Then

$$\rho = \frac{(r'^2 + r^2 + r^2\phi'^2 \sin^2\theta)^{3/2}}{((Br\phi'\sin\theta - Cr)^2 + (Cr' - Ar\phi'\sin\theta)^2 + (Ar - Br')^2)^{1/2}} \qquad \text{(A18)*}$$

where

$A = r'' - r - r\phi'^2 \sin^2\theta,$

$B = 2r' - r\phi'^2 \sin\theta \cos\theta,$

$C = r\phi'' \sin\theta + 2r'\phi' \sin\theta + 2r\phi' \cos\theta.$

4. Equation of curve given by: $r = r(\phi), \theta = \theta(\phi)$. Then

$$\rho = \frac{(r'^2 + r^2\theta'^2 + r^2 \sin^2\theta)^{3/2}}{((Br \sin\theta - Cr\theta')^2 + (Cr' - Ar \sin\theta)^2 + (Ar\theta' - Br')^2)^{1/2}}, \qquad (A19)^*$$

where

$A = r'' - r\theta'^2 - r \sin^2\theta,$

$B = r\theta'' + 2r'\theta' - r \sin\theta \cos\theta,$

$C = 2r' \sin\theta + 2r\theta' \cos\theta$

A-3. *Implicit representation of curves.*

Two dimensional curves

1. Cartesian coordinates. Equation of curve given by $F(x,y) = 0$. Then

$$\rho = \frac{(F_x^2 + F_y^2)^{3/2}}{F_x^2 F_{yy} - 2F_x F_y F_{xy} + F_y^2 F_{xx}} \qquad (A20)$$

where $F_x = \dfrac{\partial F(x,y)}{\partial x}$, etc.

2. Polar coordinates. Equation of curve given by $F(r,\theta) = 0$. Then

$$\rho = \frac{(r^2 F_r^2 + F_\theta^2)^{3/2}}{r^2 F_r^3 + r(F_r^2 F_{\theta\theta} - 2F_r F_\theta F_{r\theta} + F_\theta^2 F_{rr}) + 2F_r F_\theta^2} \qquad (A21)^*$$

where $F_r = \dfrac{\partial F(r,\theta)}{\partial r}$, etc.

Space curves.

1. Cartesian coordinates. Equation of curve defined by intersection of surface $F(x,y,z) = 0$ with surface $G(x,y,z) = 0$. Then

$$\frac{dy}{dx} = \frac{-\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} \,, \quad \frac{dz}{dx} = \frac{-\begin{vmatrix} F_y & F_x \\ G_y & G_x \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} \,. \tag{A22}$$

Equation (A22) is used to compute y'' and z''. Then ρ is computed from equation (A9).

2. Cylindrical coordinates. Equation of curve defined by intersection of surface $F(r,\theta,z) = 0$ with surface $G(r,\theta,z) = 0$. Then

$$\frac{dr}{d\theta} = \frac{-\begin{vmatrix} F_\theta & F_z \\ G_\theta & G_z \end{vmatrix}}{\begin{vmatrix} F_r & F_z \\ G_r & G_z \end{vmatrix}} \,, \quad \frac{dz}{d\theta} = \frac{-\begin{vmatrix} F_r & F_\theta \\ G_r & G_\theta \end{vmatrix}}{\begin{vmatrix} F_r & F_z \\ G_r & G_z \end{vmatrix}} \,. \tag{A23}$$

Equation (A23) is used to compute r'' and z''. Then ρ is computed from equation (A14).

3. Spherical coordinates. Equation of curve defined by intersection of surface $F(r,\theta,\phi) = 0$ with surface $G(r,\theta,\phi) = 0$. Then

$$\frac{dr}{d\theta} = \frac{-\begin{vmatrix} F_\theta & F_\phi \\ G_\theta & G_\phi \end{vmatrix}}{\begin{vmatrix} F_r & F_\phi \\ G_r & G_\phi \end{vmatrix}}, \qquad \frac{d\phi}{d\theta} = \frac{-\begin{vmatrix} F_r & F_\theta \\ G_r & G_\theta \end{vmatrix}}{\begin{vmatrix} F_r & F_\phi \\ G_r & G_\phi \end{vmatrix}} .$$

Equations (A24) are used to compute r'' and ϕ''. Then ρ is computed from equation (A18).

A-4. Discrete representation of curves.

Let $\mathbf{r}_0$, $\mathbf{r}_1$ and $\mathbf{r}_2$ be radius vectors to points on the curve with $\mathbf{r}_0$ going to the middle of the three points. Then define $\mathbf{v}_1$, $\mathbf{v}_2$, $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, r_1 and r_2 by

$$\mathbf{v}_1 = \mathbf{r}_1 - \mathbf{r}_0, \qquad \mathbf{v}_2 = \mathbf{r}_2 - \mathbf{r}_0,$$

$$\hat{\mathbf{n}}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}, \qquad \hat{\mathbf{n}}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|},$$

$$r_1 = \frac{|\mathbf{v}_1| - \hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2\,|\mathbf{v}_2|}{1-(\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2)^2}, \qquad r_2 = \frac{|\mathbf{v}_2| - \hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2\,|\mathbf{v}_1|}{1-(\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2)^2}$$

Then

$$\rho = (r_1^2 + 2r_1r_2\,\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2 + r_2^2)^{1/2}. \qquad \text{(A25)*}$$

If $|\mathbf{v}_1| = |\mathbf{v}_2|$ then reduces to

$$\rho = \frac{|\mathbf{v}_1|}{\sqrt{2(1+\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2)}} \qquad \text{(A26)*}$$

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